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On the Scaled Trace Forms and the Transfer of a Number Field Extension

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Let F be a number field. Let L/F be a field extension of degree m and let $\mathrm{Tr}_{L/F}$ be the trace map. In this paper we ask for which quadratic forms φ over F with $\dim \varphi = mn$ does exist a form ρ over L with $\dim \rho = n$ such that φ is isometric to $\mathrm{Tr}_{L/F}(\rho)$. We further show that for given $m, n, r \in \mathbb{N}$ and quadratic forms $\varphi_1, ..., \varphi_r$ over F with $\dim \varphi_i = mn$ there exist an extension L/F of degree m and $\rho_1, ..., \rho_r$ over L with $\dim \rho_i = n$ such that $\mathrm{Tr}_{L/F}(\rho_i)$ is isometric to φ_i for $i = 1, ..., r$.

1. Introduction and Main Results

Let F be a field with char $F \neq 2$. In this paper the notation (quadratic) form over F will always stand for a nondegenerate quadratic form defined over F (see [S, 1], e.g.). If two quadratic forms φ and ψ over F are isometric resp. Witt equivalent we write $\varphi \cong \psi$ resp. $\varphi = \psi$. Although we mostly work with forms instead of Witt classes, we use as invariants for a form φ the Witt class invariants, the rank rk φ, the discriminant dis φ, the Witt invariant $c(\varphi)$, and for every ordering P of F the signature $\mathrm{sgn}_P\, \varphi$ as defined in [S, 1], since it is more convenient for our purposes. Let X_F denote the set of all orderings of F. For $a \in F^*$ let $H(a)$ denote the set of all $P \in X_F$ for which $a \in P$.

If L is a finite, separable extension of F then $\mathrm{Tr}_{L/F}$ denotes the trace map and $N_{L/F}$ denotes the norm map. Let $\Lambda_{L/F} = N_{L/F}(L^*)\, F^{*2}$. For every $P \in X_F$ let $E_{L/F}(P)$ denote the set of all $Q \in X_L$ for which $Q \cap F = P$. Furthermore, for $a \in L^*$ we set $\langle L_a \rangle = \mathrm{Tr}_{L/F}(\langle a \rangle)$. A form φ over F with $\dim \varphi = [L : F]$ is called scaled trace form of L/F if there exists some $a \in L^*$ such that $\varphi \cong \langle L_a \rangle$. Note that if φ is a scaled trace form of L/F then dis $\varphi \in \Lambda_{L/F}$ (see [K, 1.1(ii)], e.g.).

A number field will always denote a finite extension of $\mathbb{Q}$. If F is a number field then R_F denotes the integral closure of $\mathbb{Z}$ in F. If $\mathfrak{p}$ is any finite

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prime spot of F then $F_{\mathfrak{p}}$ denotes the completion and $v_{\mathfrak{p}}$ denotes the corresponding valuation. For a form φ over F we set $\varphi_{\mathfrak{p}} = \varphi_{F_{\mathfrak{p}}}$. Furthermore, c will denote the local Witt invariant. In the following the notation "prime spot" will always refer to a finite prime spot of F. We call a field F local number field if it is isomorphic to the completion at some prime spot of a number field. If P is an ordering of F then F_P denotes the real closure of F with respect to P. Note that all results we shall obtain are also valid for finite, separable extensions of $k(t)$ resp. $k((t))$ where k denotes a finite field with char $k \neq 2$, and the proofs given here can be transferred immediately.

Let F be a number field and let L/F be an extension of degree m. In [L, W], Leep and Wadsworth determined the quadratic forms φ over F for which there exists a form ρ over L such that φ is Witt equivalent to $\mathrm{Tr}_{L/F}(\rho)$. Their results hold for a much larger class of fields and only little number theory was needed for the proofs. In this paper we want to examine the corresponding question for forms: For which quadratic forms φ over F with $\dim \varphi = mn$ does exist a form ρ over L with $\dim \rho = n$ such that $\mathrm{Tr}_{L/F}(\rho) \cong \varphi$? We shall see that in order to get results about this question we have to use number theoretical methods like Dirichlet's density theorem, the product formula, and results from local class field theory as in [B, 2]. We further show that for $m, n, r \in \mathbb{N}$ and forms $\varphi_1, ..., \varphi_r$ over F with $\dim \varphi_i = mn$ there exist an extension L/F of degree m and $\rho_1, ..., \rho_r$ over L with $\dim \rho_i = n$ such that $\varphi_i \cong \mathrm{Tr}_{L/F}(\rho_i)$ for $i = 1, ..., r$ (see Theorem 6).

In the first two sections we determine the scaled trace forms of a local number field extension and of a number field extension of odd degree. We begin by looking at the local case. Most of the required work has already been done in earlier paper. In Section 3 we use these results to prove:

THEOREM 1. *Let F be a number field and let L/F be an extension of odd degree m. Let φ be a form over F with* $\dim \varphi = m$. *Then φ is a scaled trace form of L/F if and only if $|\mathrm{sgn}_P \varphi| \leqslant \# E_{L/F}(P)$ for all orderings $P \in X_F$ and $c_{\mathfrak{p}}(\varphi) = 1$ for all nondyadic prime spots $\mathfrak{p}$ for which there is only one prime spot of L lying above $\mathfrak{p}$.*

Using a well known characterization of scaled trace forms (see [C, P, III 5.2; S, 2; W], e.g.) we can formulate Theorem 1 as follows:

REFORMULATION. *Let F be a number field. Let $f(t) \in F[t]$ be irreducible of odd degree m. Let $\varphi = \langle d_1, ..., d_m \rangle$ be a form over F and let $D \in \mathrm{Sym}(m, F)$ denote the corresponding diagonal matrix. Then $f(t)$ is the characteristic polynomial of a matrix BD with $B \in \mathrm{Sym}(m, F)$ if and only if $|\mathrm{sgn}_P \varphi| \leqslant \# \{zeros\ of\ f\ in\ F_P\}$ for every $P \in X_F$ and $c_{\mathfrak{p}}(\varphi) = 1$ for all nondyadic prime spots $\mathfrak{p}$ for which $f(t)$ remains irreducible in $F_{\mathfrak{p}}$.*

In particular, we get the following result shown in [B, 1, Theorem 1]. Our approach generalizes Bender's proof and to prove Theorem 1 we will need Bender's determination of the scaled trace forms of a local dyadic number field extension (see [B, 2]).

COROLLARY (Bender). *Let F be a number field and let $f(t) \in F[t]$ be an irreducible polynomial of odd degree m. Then $f(t)$ is the characteristic polynomial of a symmetric matrix $B \in \mathrm{Sym}(m, F)$ if and only if $f(t)$ is totally real; that is f splits into linear factors over every real closure.*

Unfortunately, we have not been able to determine the scaled trace forms of a number field extension of even degree. See Proposition 2. In Section 4 we apply Theorem 1 to generalize a theorem of W. Scharlau and W. C. Waterhouse (see [S, 2; W]). They consider only one form instead of finitely many.

THEOREM 2. *Let F be a number field and let $m, n \in \mathbb{N}$. Let $\varphi_1, ..., \varphi_n$ be forms over F with $\dim \varphi_i = m$ for $i = 1, ..., n$. Then there exists an extension L/F of degree m and $a_1, ..., a_n \in L^*$ such that $\langle L_{a_i} \rangle \cong \varphi_i$ for $i = 1, ..., n$.*

Let L/F be a number field extension of degree m. In the last section we examine, more generally, for which quadratic forms φ over F with $\dim \rho = mn$ there exists a form ρ over L with $\dim \rho = n$ such that $\mathrm{Tr}_{L/F}(\rho) \cong \varphi$. We shall prove:

THEOREM 3. *Let L/F be a number field extension of odd degree m. Let φ be a form over F with $\dim \varphi = mn$.*

(a) *Let $n = 2$. There exist $a_1, a_2 \in L^*$ such that $\mathrm{Tr}_{L/F}(\langle a_1, a_2 \rangle) \cong \varphi$ if and only if $|\mathrm{sgn}_P \varphi| \leqslant 2 \# E_{L/F}(P)$ for all $P \in X_F$ and for all nondyadic prime spots $\not{p}$ of F with $c_{\not{p}}(\varphi) = -1$ either $\mathrm{dis}\, \varphi \notin F_{\not{p}}^{*2}$ or there is more than one prime spot of L lying above $\not{p}$.*

(b) *Let $n \geqslant 3$. Then there exist $a_1, ..., a_n \in L^*$ such that $\mathrm{Tr}_{L/F}(\langle a_1, ..., a_n \rangle) \cong \varphi$ if and only if $|\mathrm{sgn}_P \varphi| \leqslant n \# E_{L/F}(P)$ for all $P \in X_F$.*

The case m even remains open, but at least we shall be able to solve the above problem in some special cases (see Propositions 5, 6, and 7) and further for finite local number field extensions and for extensions L/F, where L and F are finite extensions of the rational function field $R(X)$, R real closed. See Theorems 4 and 5. Finally, we give a generalized version of Theorem 2. See Theorem 6.

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2. The Scaled Trace Forms of a Local Number Field Extension

In this section we determine the scaled trace forms of a local number field extension. Recall that a Witt class over a local number field is uniquely determined by (rk, dis, c). The following four lemmas will give a complete survey of the scaled trace forms of a local number field extension.

Lemma 1. *Let F be a nondyadic local number field and let L/F be an extension of odd degree m. Then a form φ over F with* dim $\varphi = m$ *is a scaled trace form of L/F if and only if $\varphi = \langle b \rangle = (1, b, 1)$ for some $b \in F^*$.*

Proof. By [K, 1.1(iv)], every $\varphi = \langle b \rangle$ is a scaled trace form of L/F. The Witt invariant of a scaled trace form has to be 1 by [K, 2.5, 2.6]. ∎

Lemma 2. *Let F be a local number field and let L/F be an extension of even degree m. Let E denote the maximal multiquadratic extension of F contained in L. Let φ be a form over F with* dim $\varphi = m$. *Then*

(a) $\Lambda_{L/F} = \Lambda_{E/F}$.

(b) $(0, 1, 1)$ *is a scaled race form of L/F.*

(c) *If* dis $\varphi \notin F^{*2}$ *then φ is a scaled trace form of L/F if and only if* dis $\varphi \in \Lambda_{L/F}$.

Proof. (a) See [L, W, 3.2(1)]. (b) See, e.g., [K, 1.1(iv)]. (c) If dis $\varphi \in \Lambda_{L/F}$ there exists some $a \in L^*$ such that $N_{L/F}(a) = \operatorname{dis} \varphi \operatorname{dis}\langle L \rangle$ (mod F^{*2}) by [K, 1.1(i)]. Then $\langle L_a \rangle = (0, d, e)$ where $e = \pm 1$, $d = \operatorname{dis} \varphi$. Now since dis $\varphi \notin F^{*2}$ we have $(0, d, -e) = \langle b \rangle (0, d, e)$ for a suitable $b \in F^*$. ∎

For a nondyadic local number field it remains to examine for which extensions of even degree L/F the Witt class $(0, 1, -1)$ is a scaled trace form of L/F. We denote representatives of the four square classes of F^*/F^{*2} by $1, \varepsilon, \pi, \varepsilon\pi$ where π is a prime element and ε is a non-square unit.

Lemma 3. *Let F be a nondyadic local number field and let L/F be an extension of even degree m. Let E denote the maximal multiquadratic subextension of L/F. Then $(0, 1, -1)$ is a scaled trace form of L/F if and only if $E = F(\sqrt{\pi}, \sqrt{\varepsilon})$.*

Proof. Let $a \in L^*$. Let T denote the inertia field of L/F and let S/T denote the maximal tamely ramified extension contained in L. If $[L:T]$ is odd then by Lemma 1, $\langle L_a \rangle = \langle T_{a'} \rangle$ for a suitable $a' \in T^*$. From [C, P, II 3.2] we conclude $\langle T_{a'} \rangle \neq (0, 1, -1)$. Now assume $s = [S:T]$ is even. Then $S = T(\beta)$, where $\beta = \sqrt[s]{\pi u}$ for a suitable unit $u \in T^*$. By Lemma 1,

$\mathrm{Tr}_{L/S}(\langle a\rangle)=\langle a'\rangle$ for some $a'\in S^*$. If $a'=\beta$ or $a'=\beta\varepsilon_T \pmod{S^{*2}}$ then by [K, 1.7], $\mathrm{Tr}_{S/T}(\langle a'\rangle)=0$ and hence $\langle L_a\rangle=0$. If $a'=1$ resp. $a'=\varepsilon_T$ $\pmod{S^{*2}}$ then we get from [K, 1.7], $\mathrm{Tr}_{S/T}(\langle a'\rangle)=\langle s, s\pi u\rangle$ resp. $\mathrm{Tr}_{S/T}(\langle a'\rangle)=\langle s\varepsilon_T\rangle\langle 1, \pi u\rangle$. If $[T:F]$ is odd, we see that $\mathrm{dis}\langle L_a\rangle\notin F^{*2}$. If $u=\varepsilon_T \pmod{T^{*2}}$ and $[T:F]$ is even then by [C, P, II 3.2], $\mathrm{dis}\langle L_a\rangle_F=N_{T/F}(\varepsilon_T)\,\mathrm{dis}\langle T\rangle_F=\varepsilon \pmod{F^{*2}}$. Thus, if $[T:F]$ is odd or $T(\sqrt{\pi})$ is not contained in L then $\langle L_a\rangle\neq(0, 1, -1)$. If $[T:F]$ is even and u is a square in T then one sees immediately, using [C, P, II 3.2] and the above computations, that $(0, 1, -1)$ is a scaled trace form of L/F. ∎

LEMMA 4. (Bender). *Let F be a dyadic number field and let L/F be an extension of degree $m\geqslant 3$.*

(i) *If m is odd, every form φ over F with* $\dim\varphi=m$ *is a scaled trace form of L/F.*

(ii) *Assume $F=\mathbb{Q}_2$, $m=4$, and there exists no quadratic extension E/F contained in L. Then a 4-dimensional form φ is a scaled trace form of L/F if and only if $\varphi\neq(0, 1, -1)$.*

(iii) *Assume m is even and if $m=4$ and $F=\mathbb{Q}_2$ then there exists a quadratic extension E/F contained in L. Then $(0, 1, -1)$ is a scaled trace form of L/F.*

Proof. (i) See [B, 2, Lemma 3]. If L/F contains no quadratic subextension then $\Lambda_{L/F}=F^*/F^{*2}$ by Lemma 2(a). Thus, (ii) and (iii) follow from Lemma 2 and [B, 2, Theorem 1]. ∎

For further applications we state finally:

LEMMA 5. *Let F be a local number field. If L/F is an extension of even degree then there exists some $a\in L^*$ such that $c(\langle L_a\rangle)=-1$.*

Proof. If $\Lambda_{L/F}$ contains a square class other than F^{*2} apply Lemma 2(c). If F is nondyadic and $|\Lambda_{L/F}|=1$ then by Lemma 3, $(0, 1, -1)$ is a scaled trace form of L/F. If F is dyadic apply Lemma 4. ∎

3. THE SCALED TRACE FORMS OF A NUMBER FIELD EXTENSION

In the following F will always denote a number field. We shall be able to determine the scaled trace forms of every odd degree extension. It seems to be fairly difficult to extend the results to extensions of even degree. Let L/F be a finite extension. We say that a prime spot $\mathfrak{p}$ of F splits (over L) if there is more than one prime spot of L lying above $\mathfrak{p}$. Note that for $a\in L^*$ we have $\langle L_a\rangle_{\mathfrak{p}}\cong\bigoplus_{\mathfrak{P}|\mathfrak{p}}\langle(L_{\mathfrak{P}})_a\rangle_{F_{\mathfrak{p}}}$.

LEMMA 6. *Let L/F be an extension of odd degree m. Let $\mathfrak{p}$ be a prime spot of F which splits over L. Let φ be a form over F with* $\dim \varphi = m$. *Then there exists some $a \in L^*$ such that $\langle L_a \rangle_{\mathfrak{p}} \cong \varphi_{\mathfrak{p}}$.*

Proof. Let $d = \operatorname{dis} \varphi$. Let us first assume there are two prime spots $\mathscr{P}$, $\mathscr{Q}$ lying above $\mathfrak{p}$. Assume wlog that $[L_{\mathscr{P}}:F_{\mathfrak{p}}]$ is odd. Then by Lemma 5, since $[L_{\mathscr{Q}}:F_{\mathfrak{p}}]$ is even, there exists some $a \in L_{\mathscr{Q}}^*$ such that $\langle (L_{\mathscr{Q}})_a \rangle = (0, b, -1)$ for a suitable $b \in F^*$. By Lemma 2, $(0, b, 1)$ is also a scaled trace form of $L_{\mathscr{Q}}/F_{\mathfrak{p}}$. Now, $\langle db \rangle = (1, db, 1)$ is a scaled trace form of $L_{\mathscr{P}}/F_{\mathfrak{p}}$. Using the approximation theorem we find $a, a' \in L^*$ such that $\langle L_a \rangle_{\mathfrak{p}} = (1, db, 1) + (0, b, -1)$ and $\langle L_{a'} \rangle_{\mathfrak{p}} = (1, db, 1) + (0, b, 1)$. Since the two forms have different Witt invariants one of them must be isometric to $\varphi_{\mathfrak{p}}$. If there are more than two prime ideals lying above $\mathfrak{p}$ the proof is similar and is left to the reader. ∎

PROPOSITION 1. *Let L/F be an extension of degree m and let φ be a form over F with* $\dim \varphi = m$. *Let $d \in R_F$ represent the square class* $\operatorname{dis} \varphi$. *Let Ω be a finite set of prime spots of F containing all prime spots which divide $2d$ or which are ramified in L/F. Furthermore, let $\mathscr{H}$ be a subset of X_L. Assume that for all $\mathfrak{p} \in \Omega$ there exists some $a(\mathfrak{p}) \in L^*$ satisfying $\langle L_{a(\mathfrak{p})} \rangle_{\mathfrak{p}} \cong \varphi_{\mathfrak{p}}$. Then there exists some $a \in L^*$ and a prime spot $\mathscr{Q} \notin \Omega$ of F such that*

(i) $H(a) = \mathscr{H}$.

(ii) $\langle L_a \rangle_{\mathfrak{p}} \cong \varphi_{\mathfrak{p}}$ *for every* $\mathfrak{p} \in \Omega$.

(iii) $c_{\mathfrak{p}}(\langle L_a \rangle) = 1$ *and* $\operatorname{dis}\langle L_a \rangle$ *is a unit in* $F_{\mathfrak{p}}$ $(\operatorname{mod} F_{\mathfrak{p}}^{*2})$ *for every* $\mathfrak{p} \notin \Omega$, $\mathfrak{p} \neq \mathscr{Q}$.

Proof. Almost the same statement has been proved by Bender in [B, 1, Theorem 3] and a proof of the proposition is analogous to that of [B, 1]: By the (strong) approximation theorem and hypothesis, we find some $a \in R_L$ satisfying (i) and (ii). Then applying Dirichlet's density theorem ([Lg, VIII Corollary p. 166.), we may assume a to be a unit at every prime spot $\mathscr{P}$ of L, $\mathscr{P} \cap R_F \notin \Omega$ except at one Q' of L. Then the second residue class form of $\langle L_a \rangle_{\mathfrak{p}}$ is zero for all $\mathfrak{p} \neq \mathscr{Q} := \mathscr{Q}' \cap R_F$, $\mathfrak{p} \notin \Omega$. ∎

Proof of Theorem 1. By [K, 1.1(iii)] and Lemma 1, every scaled trace form of L/F satisfies the conditions stated in the theorem. Conversely, assume φ satisfies the conditions stated in Theorem 1. Let $d \in R_F$ represent the square class $\operatorname{dis} \varphi$. Let Ω denote the set consisting of all prime spots $\mathfrak{p}$ of F which divide $2d$ or which are ramified in L/F or for which $c_{\mathfrak{p}}(\varphi) = -1$. Let $\mathscr{H}$ be a subset of X_L such that $\# \mathscr{H} \cap E_{L/F}(P) = \frac{1}{2}(\operatorname{sgn}_P \varphi + \# E_{L/F}(P))$ for all $P \in X_F$. By Lemmas 4 and 6 we can apply Proposition 1. There exists some $a \in L^*$ and a nondyadic prime spot $\mathscr{Q}$ of F such that $\langle L_a \rangle_{\mathfrak{p}} \cong \varphi_{\mathfrak{p}}$ for every $\mathfrak{p} \in \Omega$, $c_{\mathfrak{p}}(\langle L_a \rangle) = 1$ and $\operatorname{dis}\langle L_a \rangle$ is a unit in $F_{\mathfrak{p}}$ (mod $F_{\mathfrak{p}}^{*2}$) for every finite prime spot $\mathfrak{p} \notin \Omega$, $\mathfrak{p} \neq \mathscr{Q}$ and further

$$\# E_{L/F}(P) \cap H(a) - \# E_{L/F}(P) \cap H(-a) = \operatorname{sgn}_P \varphi$$

for every ordering $P \in X_F$. Thus, $\operatorname{sgn}_P \varphi = \operatorname{sgn}_P \langle L_a \rangle$ for all $P \in X_F$. Let $\delta \in R_F$ represent the square class $\operatorname{dis}\langle L_a \rangle\, d$. Then $\delta \in P$ for every $P \in X_F$, $\delta \in F_{\mathfrak{p}}^{*2}$ for $\mathfrak{p} \in \Omega$, and δ is a unit in $F_{\mathfrak{p}}$ (mod $F_{\mathfrak{p}}^{*2}$) for every $\mathfrak{p} \notin \Omega$, $\mathfrak{p} \notin \mathscr{Q}$. This shows that the forms φ and $\langle L_{a\delta} \rangle$ have the same invariants since by Hilbert's reciprocity theorem $c_{\mathscr{Q}}(\varphi) = c_{\mathscr{Q}}(\langle L_{a\delta} \rangle)$. ∎

Remark 1. Let L/F be an extension of odd degree. Let φ be a scaled trace form of L/F. For every $P \in X_F$ let $s_P := \frac{1}{2}(\# E_{L/F}(P) + \operatorname{sgn}_P \varphi)$ and let $Q_1, ..., Q_{s_P}$ denote some extensions of P to L. Then the above proof shows that it is possible to find some $a \in L^*$ with $H(a) \cap E_{L/F}(P) = \{Q_1, ..., Q_{s_P}\}$ for every $P \in X_F$ such that $\langle L_a \rangle \cong \varphi$.

Remark 2. (a) Let L/F be an extension of odd degree m. Theorem 1 implies that a form φ over F with $\dim \varphi = m$ is a scaled trace form of L/F if and only if $\varphi_{\mathfrak{p}}$ is a matrix from $L \otimes F_{\mathfrak{p}}$ to $F_{\mathfrak{p}}$ (see [B, 1]) for every (real or finite) prime spot $\mathfrak{p}$ of F. Such a local–global principle cannot hold for extensions of even degree. Leep and Wadsworth have mentioned in [L, W, 4.6] that, in general, $\Lambda_{L/F}$ does not contain $\{c \in F^* \mid c \in \Lambda_{L_{\mathscr{P}}/F_{\mathfrak{p}}}$ for all $\mathscr{P}$ of L, $\mathfrak{p}$ of F, $\mathscr{P} \mid \mathfrak{p}\}$. Moreover, a result similar to Lemma 6 is not valid for extensions L/F of even degree. For instance, if there are exactly two prime spots $\mathscr{P}_1, \mathscr{P}_2$ lying above some nondyadic prime spot $\mathfrak{p}$ such that $[L_{\mathscr{P}_i} : F_{\mathfrak{p}}] \equiv 1(2)$ for $i = 1, 2$ then $\langle L_a \rangle_{\mathfrak{p}} \neq (0, 1, -1)$ for all $a \in L^*$.

(b) By [K, 1.5], the scaled trace forms of a quadratic extension L/F are $\langle 1, -1 \rangle$ and $\langle a, b \rangle$ where $a, b \in F^*$, $-ab \notin F^{*2}$, and $-ab \in \Lambda_{L/F}$.

From Lemma 5 and Proposition 1 we conclude:

PROPOSITION 2. *Let L/F be an extension of even degree m. Let φ be a form over F such that $|\operatorname{sgn}_P \varphi| \leqslant \# E_{L/F}(P)$ for all $P \in X_F$. Then there exists some $a \in L^*$ such that $\operatorname{sgn}_P \langle L_a \rangle = \operatorname{sgn}_P \varphi$ for all $P \in X_F$ and $c(\langle L_a \rangle) = c(\varphi)$.*

Note that it is also possible to determine the scaled trace forms of an extension L/F where F is a finite extension of the rational function field $R(X)$, R real closed. See [K, 3.6].

4. Systems of Scaled Trace Forms

As an application of Theorem 1 we now want to prove Theorem 2.

Proposition 3. *Let $m, n \in \mathbb{N}$, m odd. Let $\varphi_1, ..., \varphi_n$ be forms over F with $\dim \varphi_i = m$ for $i = 1, ..., n$. Then there exists an extension L/F of degree m such that there are $a_1, ..., a_n \in L^*$ with $\langle L_{a_i} \rangle \cong \varphi_i$ for $i = 1, ..., n$. Moreover, we may assume $E_{L/F}(P) \subset H(a_1)$ for all $P \in X_F$ for which $|\operatorname{sgn}_P \varphi_i| \leqslant \operatorname{sgn}_P \varphi_1$ for $i \geqslant 2$.*

Proof. Let Δ denote the set of $P \in X_F$ for which $|\operatorname{sgn}_P \varphi_i| \leqslant \operatorname{sgn}_P \varphi_1$ for $i \geqslant 2$. Choose a polynomial $f(t) = t^m + b_{m-1} t^{m-1} + \cdots b_0 \in F[t]$ with integral coefficients, which has m distinct roots in some algebraic closure, such that the number of zeros of f in F_P equals $\operatorname{sgn}_P \varphi_1$ if $P \in \Delta$ and equals m if $P \notin \Delta$. Surely, it is possible to choose $f(t)$ as a product of polynomials of degree 1 or 2 by the approximation theorem. Let Ω be the set consisting of all finite, nondyadic prime spots $\mathfrak{p}$ of F for which $c_{\mathfrak{p}}(\varphi_i) = -1$ for some $i = 1, ..., n$. Furthermore, let $\mathscr{P} \notin \Omega$ be another finite prime spot of F. Using the approximation theorem we replace the coefficients b_i of f by $b_i c_i$ where $c_i \in F$ are sufficiently close to 1 for all $P \in X_F$ such that the number of zeros in F_P does not change and further

$$\begin{aligned}
b_i c_i &\in \mathfrak{p} \qquad \text{for} \quad \mathfrak{p} \in \Omega \cup \{\mathscr{P}\}, m-2 \geqslant i \geqslant 1,\\
b_0 c_0 &\in \mathfrak{p} \qquad \text{for} \quad \mathfrak{p} \in \Omega,\\
b_0 c_0 &\in \mathscr{P} - \mathscr{P}^2,\\
b_{m-1} c_{m-1} &\equiv 1(\mathfrak{p}) \qquad \text{for} \quad \mathfrak{p} \in \Omega,\\
b_{m-1} c_{m-1} &\in \mathscr{P}.
\end{aligned}$$

Then, the polynomial $\bar{f}$ we get is irreducible since it is an Eisenstein polynomial at $\mathscr{P}$. Set $L := F[t]/\bar{f}(t)$. Then every $\mathfrak{p} \in \Omega$ splits in L, since $\bar{f}(t) = t^{m-1}(t-1) \pmod{\mathfrak{p}}$ for $\mathfrak{p} \in \Omega$, and $\# E_{L/F}(P) = \operatorname{sgn}_P \varphi_1$ for $P \in \Delta$. Thus by Theorem 1, the forms φ_i are scaled trace forms for L/F. ∎

Proof of Theorem 2. Let $m = 2^r w$. We proceed by induction on r. Let $r \geqslant 1$. Let $\varphi_i = \varphi_i' \perp \varphi_i''$ where $\dim \varphi_i' = \dim \varphi_i'' = 2^{r-1} w$. Applying the induction hypothesis we find an extension L/F of degree $2^{r-1} w$ such that there exist $a_i, b_i \in L^*$ with $\langle L_{a_i} \rangle \cong \varphi_i'$ and $\langle L_{b_i} \rangle \cong \varphi_i''$ for $i = 1, ..., n$. By [K, Theorem 4.2], there exists a quadratic extension M/L such that $-a_i b_i \in \Lambda_{M/L}$ for $i = 1, ..., n$. Hence by [K, 1.5], there exist $c_i \in M^*$ such that $\operatorname{Tr}_{M/L}(\langle c_i \rangle) \cong \langle a_i, b_i \rangle$; that is $\operatorname{Tr}_{M/F}(\langle c_i \rangle) \cong \varphi_i$ for $i = 1, ..., n$. ∎

Remark 3. Theorem 2 generalizes a theorem of Scharlau and Waterhouse (see [S, 2; W]). The above proof does not use Hilbert's irreducibility theorem, which was an essential tool in [S, 2; W]. In order to show that every positive form over a number field F is Witt equivalent to a trace form of some L/F (see [S, 2]) one could use Proposition 3 and [K, S, Theorem 2, Corollary 4]. Note that Theorem 2 of [K, S] can be proved for number fields without Hilbert's irreducibility theorem. See [K, S; E, H, P].

5. The Transfer of Quadratic Forms

Let L/F be a number field extension of degree m. It is well known that if m is odd, for every Witt class φ over F there exists a Witt class ψ over L such that $\mathrm{Tr}_{L/F}(\psi) = \varphi$. If n is even, the question which Witt classes over F are in the image of the transfer map has been answered by Leep and Wadsworth. See [L, W, Theorem 1.13]. But apparently there are no results known about the corresponding question for forms (except [A, Satz 2.4]): For which forms φ over F with $\dim \varphi = nm$ does exist a form ψ over L with $\dim \psi = n$ such that $\mathrm{Tr}_{L/F}(\psi) \cong \varphi$? Note that if $\mathrm{Tr}_{L/F}(\psi) \cong \varphi$ then $\operatorname{dis} \varphi = N_{L/F}(\operatorname{dis} \psi)$ if $\dim \psi \equiv 0(2)$ and $\operatorname{dis} \varphi = \operatorname{dis} \mathrm{Tr}_{L/F}(\langle 1 \rangle)\, N_{L/F}(\operatorname{dis} \psi)$ if $\dim \psi \equiv 1(2)$. In particular, $\operatorname{dis} \varphi \in \Lambda_{L/F}$. In [A, Satz 2.4(2)], Arason answered the question for a quadratic extension of an arbitrary field F with $\operatorname{char} F \neq 2$. Next, we want to prove Theorem 3 which together with Theorem 1 answers the question for number field extensions of odd degree. Let $W(F)$ denote the Wittring of F and let $I(F)$ be the augmentation ideal. We shall need furthermore:

Proposition 4. *Let $F \subset L$ be two number fields. Let $a \in L^* - L^{*2}$. Then there exist infinitely many prime spots $\mathfrak{p}$ of L such that $a \notin L_{\mathfrak{p}}^{*2}$ and $\mathfrak{p} \cap R_F$ splits (completely) in L.*

Proof. By the global square theorem there exist infinitely many prime spots $\mathfrak{p}$ of L such that $a \notin L_{\mathfrak{p}}^{*2}$. Choose a nondyadic $\mathcal{Q}$ with $a \notin L_{\mathcal{Q}}^{*2}$. By the (strong) approximation theorem there exists some $b \in R_L$ such that $b \in P$ for all $P \in X_L$, $b \equiv 1(4\mathfrak{p})$ for all dyadic prime spots $\mathfrak{p}$ of L, $(a, b)_{\mathcal{Q}} = -1$, and $b \equiv 1(\mathfrak{p})$ for all nondyadic $\mathfrak{p} \neq \mathcal{Q}$ which are ramified in $L/\mathbb{Q}$ or for which $v_{\mathfrak{p}}(a) \equiv 1(2)$. Let Ω denote the set consisting of all prime spots $\mathfrak{p}$ of L which are dyadic or ramified in $L/\mathbb{Q}$ or for which $v_{\mathfrak{p}}(a) \equiv 1(2)$ or $\mathfrak{p} = \mathcal{Q}$. Define the cycle $C = \prod_{\mathfrak{p} \in \Omega} \mathfrak{p}(4) \prod_{P \in X_L} P$ (see [Lg, VI]). Let P_C denote the set of all $s \in L$ with $s \equiv 1 \pmod{C}$. Let $w(\mathcal{Q})$ denote the maximal number w such that $\mathcal{Q}^w$ divides (b). The ideal $(b)/\mathcal{Q}^{w(\mathcal{Q})}$ is relatively prime to C. By Dirichlet's density theorem [Lg, VIII, Corollary p. 166]) there exist infinitely many prime ideals $\mathcal{R}$ of L such that $(bs) = \mathcal{R}\mathcal{Q}^{w(\mathcal{Q})}$ for some $s \in P_C$ and furthermore $\deg \mathcal{R} = [L_{\mathcal{R}} : \mathbb{Q}_q] = 1$ for $(q) = \mathbb{Z} \cap \mathcal{R}$. Since $\mathcal{R}$ is unramified in $L/\mathbb{Q}$, q splits completely in L; that is there are $[L : \mathbb{Q}]$ prime spots in L lying above q. Thus, $\mathscr{R} \cap R_F$ splits completely in L. By construction, we have $(a, bs)_{\mathfrak{p}} = 1$ for all $\mathfrak{p} \neq \mathfrak{Q}, \mathscr{R}$ and $(a, bs)_{\mathfrak{Q}} = -1$. Hence, by Hilbert's reciprocity theorem $(a, bs)_{\mathscr{R}} = -1$ and therefore $a \notin L^{*2}_{\mathscr{R}}$. ▮

Let L/F be a finite extension and let $\varphi \in I^2(F)$ be a torsion form. Then by a result of Leep and Wadsworth (see [K, 1.2])], there exists a torsion form $\rho \in I^2(L)$ such that $\mathrm{Tr}_{L/F}(\rho) = \varphi$. This also easily follows from Milnor's result [S, 1, 6.4.4]: The torsion form φ is Witt equivalent to a quaternion form $\langle 1, a, b, ab \rangle$. For every prime spot $\mathfrak{p}$ satisfying $c_{\mathfrak{p}}(\varphi) = -1$ choose one prime spot $\mathfrak{p}'$ of L lying above $\mathfrak{p}$. Then there exists a quaternion form ρ over L which has nontrivial Witt invariant exactly at all the prime spots $\mathfrak{p}'$ and hence is a torsion form. Then by [S, 1, 6.4.4], $\varphi = \mathrm{Tr}_{L/F}(\rho)$.

LEMMA 7. *Let L/F be a finite extension and let $\varphi \in I(F)$ be a torsion form. Then there exists a torsion form $\rho \in I(L)$ such that $\mathrm{Tr}_{L/F}(\rho) = \varphi$ if and only if there exists a sum of squares $a \in L^*$ such that $N_{L/F}(a) = \mathrm{dis}\,\varphi$ (mod F^{*2}).*

Proof. If $\mathrm{Tr}_{L/F}(\rho) = \varphi$ for a torsion form $\rho \in I(L)$ then $\mathrm{dis}\,\rho$ is a sum of squares and $N_{L/F}(\mathrm{dis}\,\rho) = \mathrm{dis}\,\varphi$ (mod F^{*2}). If there exists a sum of squares $a \in L^*$ such that $N_{L/F}(a) = \mathrm{dis}\,\varphi$ then $\psi := \varphi - \mathrm{Tr}_{L/F}(\langle 1, -a \rangle)$ is a torsion form in $I^2(F)$. As mentioned above, there exists a torsion form $\rho \in I^2(L)$ with $\mathrm{Tr}_{L/F}(\rho) = \psi$; that is $\mathrm{Tr}_{L/F}(\rho + \langle 1, -a \rangle) = \varphi$. ▮

COROLLARY. *Let L/F be an extension of odd degree. Then for every torsion form $\varphi \in I(F)$ there exists a torsion form $\rho \in I(L)$ such that $\mathrm{Tr}_{L/F}(\rho) = \varphi$.*

Proof. If φ is a torsion form over F then $\mathrm{dis}\,\varphi$ is a sum of squares and $N_{L/F}(\mathrm{dis}\,\varphi) = \mathrm{dis}\,\varphi$ (mod F^{*2}). ▮

Remark 4. A quadratic form over F of dimension n can be uniquely defined as follows. Let n be even. For every $P \in X_F$ choose some even $s_P \in \mathbb{Z}$, $|s_P| \leqslant n$, choose $d \in F^*$ such that $d \in -P$ if and only if $s_P \equiv 2(4)$ for $P \in X_F$ and an odd (even) number of prime spots $\mathfrak{p}_1, ..., \mathfrak{p}_r$ if the number of $P \in X_F$ satisfying $s_P \equiv 4, 6(8)$ is odd (even). If n is odd choose for every $P \in X_F$ some odd $s_P \in \mathbb{Z}$, $|s_P| \leqslant n$, choose $d \in F^*$ such that $d \in -P$ if and only if $s_P - 1 \equiv 2(4)$ for $P \in X_F$ and an odd (even) number of prime spots $\mathfrak{p}_1, ..., \mathfrak{p}_r$ if the number of $P \in X_F$ satisfying $s_P \equiv 3, 5(8)$ is odd (even). Then there exists one and only one form φ over F with $\dim \varphi = n$, $\mathrm{sgn}_P\,\varphi = s_P$ for $P \in X_F$, $\mathrm{dis}\,\varphi = d$, and $c_{\mathfrak{p}}(\varphi) = -1$ if and only if $\mathfrak{p} \in \{\mathfrak{p}_1, ..., \mathfrak{p}_r\}$.

Proof of Theorem 3. The conditions for the signature values of φ in (a) and (b) are necessary by [S, 1, 3.4.5]. Assume $\mathrm{Tr}_{L/F}(\langle a_1, a_2 \rangle) \cong \varphi$ for

$a_1, a_2 \in L^*$. Assume $\mathfrak{p}$ is a nondyadic prime spot of F which does not split in L and $\operatorname{dis} \varphi \in F_{\mathfrak{p}}^{*2}$. Then by Lemma 1, $c_{\mathfrak{p}}(\varphi) = (\operatorname{dis}\langle L_{a_1}\rangle, \operatorname{dis}\langle L_{a_2}\rangle)_{\mathfrak{p}} = 1$, since $\operatorname{dis}\langle L_{a_2}\rangle = -\operatorname{dis} \varphi \operatorname{dis}\langle L_{a_1}\rangle$.

Now assume φ satisfies the conditions of (a). Let $d \in F^*$ represent the square class $\operatorname{dis} \varphi F^{*2}$. Let Ω denote the set of all nondyadic prime spots $\mathfrak{p}$ of F which do not split in L and for which furthermore $c_{\mathfrak{p}}(\varphi) = -1$ and $d \notin F_{\mathfrak{p}}^{*2}$. We now define two forms φ_1, φ_2 over F of dimension m. Set $\operatorname{sgn}_P \varphi_1 = \frac{1}{2} \operatorname{sgn}_P \varphi$ if $\operatorname{sgn}_P \varphi \equiv 2(4)$ and $\operatorname{sgn}_P \varphi_1 = \frac{1}{2} \operatorname{sgn}_P \varphi - 1$ if $\operatorname{sgn}_P \varphi \equiv 0(4)$. Set $\operatorname{sgn}_P \varphi_2 = \operatorname{sgn}_P \varphi - \operatorname{sgn}_P \varphi_1$. Note that $|\operatorname{sgn}_P \varphi_i| \leqslant \# E_{L/F}(P)$ for every $P \in X_F$, $i = 1, 2$, since if $\operatorname{sgn}_P \varphi \equiv 0(4)$ then $|\operatorname{sgn}_P \varphi| < 2\# E_{L/F}(P)$. Choose a nondyadic prime spot $\mathfrak{Q}$ which splits in L such that $c_{\mathfrak{Q}}(\varphi) = 1$. If $d \notin F^{*2}$ choose $\mathfrak{Q}$ such that furthermore $d \notin F_{\mathfrak{Q}}^{*2}$. This is possible by Proposition 4, since if $d \notin F^{*2}$ then also $d \notin L^{*2}$. Set $c_{\mathfrak{p}}(\varphi_1) = c_{\mathfrak{p}}(\varphi_2) = 1$ if $\mathfrak{p} \neq \mathfrak{Q}, c_{\mathfrak{p}}(\varphi) = 1$ or $\mathfrak{p} \in \Omega$. If $\mathfrak{p} \notin \Omega$ and $c_{\mathfrak{p}}(\varphi) = -1$ set $c_{\mathfrak{p}}(\varphi_1) = -1$ and $c_{\mathfrak{p}}(\varphi_2) = 1$. Furthermore, for $i = 1, 2$ set $c_{\mathfrak{Q}}(\varphi_i) = 1(-1)$ if $\# \{\mathfrak{p}, P \mid \mathfrak{p} \neq \mathfrak{Q}, c_{\mathfrak{p}}(\varphi_i) = -1, P \in X_F, c_P(\varphi_i) = -1\}$ is even (odd).

Using the strong approximation theorem and Dirichlet's density theorem as above we find some $a \in R_F$ such that $a \in -P$ if and only if $\operatorname{sgn}_P \varphi_1 - 1 \equiv 2(4)$ for $P \in X_F$, $a \equiv 1(4\mathfrak{p})$ for all dyadic prime spots, $a \equiv 1(\mathfrak{p})$ for all nondyadic $\mathfrak{p} \neq \mathfrak{Q}$ satisfying $v_{\mathfrak{p}}(d) \equiv 1(2), (a, d)_{\mathfrak{p}} = -1$ for $\mathfrak{p} \in \Omega$, further $(a, d)_{\mathfrak{Q}} = 1(-1)$ if $\# \mathscr{X} := \Omega \cup \{P \in X_F \mid d \in -P, a \in -P\}$ is even (odd) and a is a unit at all other prime spots except at one $\mathscr{R}$. Note that if $\# \mathscr{X}$ is odd then $d \notin F^{*2}$. By Hilbert's reciprocity theorem, $(a, d)_{\mathscr{R}} = 1$. Set $\operatorname{dis} \varphi_1 = a$ and $\operatorname{dis} \varphi_2 = -ad$. By Remark 4, φ_1 is well defined. In order to show that φ_2 is well defined we have to check $ad \in P$ iff $\operatorname{sgn}_P \varphi - \operatorname{sgn}_P \varphi_1 - 1 \equiv 2(4)$ for every $P \in X_F$. If $d \in P$ then $\operatorname{sgn}_P \varphi \equiv 0(4)$ and $-ad \in P$ if and only if $\operatorname{sgn}_P \varphi_1 \equiv 3(4)$. If $-d \in P$ then $\operatorname{sgn}_P \varphi \equiv 2(4)$ and $-ad \in P$ if and only if $\operatorname{sgn}_P \varphi_1 \equiv 1(4)$. Hence, also φ_2 is well defined. By construction and Theorem 1, both forms are scaled trace forms of L/F. Thus there are $a_1, a_2 \in L^*$ such that $\operatorname{Tr}_{L/F}(\langle a_1, a_2\rangle) = \varphi_1 + \varphi_2$. It remains to show $\varphi \cong \varphi_1 + \varphi_2$. For this, we have to check $c(\varphi_1 + \varphi_2) = c(\varphi_1)\, c(\varphi_2)(a, d)_F = c(\varphi)$. This holds locally for all $\mathfrak{p} \neq \mathfrak{Q}$ and hence it holds globally.

Proof of (b). Let φ be a form over F with $\dim \varphi = mn$, $n \geqslant 3$, $|\operatorname{sgn}_P \varphi| \leqslant n\# E_{L/F}(P)$ for $P \in X_F$. Then there exists some $\rho \in W(F)$ with $\operatorname{rk} \rho \equiv n(2)$, $|\operatorname{sgn}_Q \rho| \leqslant n$ for all $Q \in X_L$, and $\sum_{Q \in E_{L/F}(P)} \operatorname{sgn}_Q \rho = \operatorname{sgn}_P \varphi$ for all $P \in X_F$. Thus by the corollary above and [S, 1, 3.4.5], there exists a torsion form $\psi \in I(L)$ such that $\operatorname{Tr}_{L/F}(\psi) = \operatorname{Tr}_{L/F}(\rho) - \varphi$. Let τ be a form of dimension n representing the Witt class $\rho - \psi$. Then $\operatorname{Tr}_{L/F}(\tau) \cong \varphi$. ∎

Unfortunately, the arguments used in the proof of (b) do not work in the case $n \leqslant 2$. But they can also be used in the case of an extension of even degree and $n \geqslant 3$:

PROPOSITION 5. *Let L/F be an extension of even degree m.*

(a) *For every sum of squares $d \in \Lambda_{L/F}$ there exists a sum of squares $a \in L^*$ with $N_{L/F}(a) = d$ $(\bmod F^{*2})$ if and only if for every torsion form $\varphi \in I(F)$ with $\operatorname{dis} \varphi \in \Lambda_{L/F}$ there exists a torsion form $\rho \in I(L)$ with $\operatorname{Tr}_{L/F}(\rho) = \varphi$.*

(b) *Assume the equivalent conditions in* (a) *hold for L/F. Then for every form ψ over F with $\dim \psi = mn$, $n \geqslant 3$, there exists a form ρ over L with $\dim \rho = n$ and $\operatorname{Tr}_{L/F}(\rho) \cong \psi$ if and only if $\operatorname{dis} \psi \in \Lambda_{L/F}$ and $|\operatorname{sgn}_P \psi| \leqslant n \# E_{L/F}(P)$ for $P \in X_F$.*

Proof. (a) is only a reformulation of Lemma 7 for even m. To prove (b) use the same arguments as in the proof of Theorem 3(b). ∎

PROPOSITION 6 (Arason). *Let $L = F(\sqrt{d})/F$ be a quadratic extension. Let φ be a form over F with $\dim \varphi = 2n$. Then the following statements are equivalent*:

(i) *There exists a form ρ over L with $\dim \rho = n$ such that $\operatorname{Tr}_{L/F}(\rho) \cong \varphi$.*

(ii) $\varphi\langle 1, -d\rangle = 0$.

(iii) $\operatorname{dis} \varphi \in \Lambda_{L/F}$ *and* $|\operatorname{sgn}_P \varphi| \leqslant n \# E_{L/F}(P)$ *for all* $P \in X_F$.

Proof. [A, Satz 2.4(2)]. ∎

PROPOSITION 7. *Let L/F be a finite extension.*

(a) *Suppose every $P \in X_F$ has at most two extensions to L. Then L/F satisfies the equivalent conditions of Proposition* 5(a).

(b) *Suppose L/F is a Galois extension. Then for every sum of squares $t \in F^*$ for which there exists some $a \in L^*$ with $N_{L/F}(a) = t$ there exists a sum of squares $a' \in L^*$ with $N_{L/F}(a') = t$.*

Proof. (a) Let $t \in F^*$ be a sum of squares such that there exists some $a \in L^*, z \in F^*$ with $N_{L/F}(a) = tz^2$. If $Q \cap F = P \cap F$ for $Q, P \in X_L$ then $a \in Q, P$ or $-a \in Q, P$, since $\operatorname{dis}\langle L_a\rangle = \operatorname{dis} \operatorname{Tr}_{L/F}(\langle 1\rangle)\, t$ $(\bmod F^{*2})$ (apply [S, 1, 3.4.5]). Let $\mathscr{H}$ denote the set of all $P \in X_F$ satisfying $E_{L/F}(P) \cap H(a) = \varnothing$. Then there exists some $b \in F^*$ with $H(-b) = \mathscr{H}$. Then ab is a sum of squares in L and $N_{L/F}(ab) = b^{[L:F]} z^2 t$.

(b) Let G be the Galois group of L/F. Assume t is a sum of squares in F such that there exists some $a \in L^*$ with $N_{L/F}(a) = t$. We claim that we may assume wlog that for all $P \in X_F$ either $a \in Q$ for all $Q \in E_{L/F}(P)$ or $a \in -Q$ for exactly one $Q \in E_{L/F}(P)$. Suppose there are $Q, Q' \in E_{L/F}(P)$ with $a \in -Q, -Q', Q \neq Q'$ for $P \in X_F$. There exists some $\sigma \in G$ such that

$Q' = \sigma(Q)$. Choose some $b \in L^*$ such that $b \in -Q$ and $b \in R$ for all $R \in X_L$, $R \neq Q$. Then $\sigma(b) \in -Q'$ and $\sigma(b) \in R$ for all $R \neq Q'$. We have $N_{L/F}(ab\sigma(b^{-1})) = t$. Thus continuing this procedure proves the claim.

Assume now that for $P \in X_F$ we have $a \in -Q \in E_{L/F}(P)$ and $a \in Q'$ for all $Q' \in E_{L/F}(P)$, $Q' \neq Q$. Then for every $\sigma \in G$ we have $\sigma(a) \in -\sigma(Q)$ and $\sigma(a) \in Q'$ for $Q' \neq \sigma(Q)$, $Q' \in E_{L/F}(P)$. Then we have $\prod_{\sigma \neq 1} \sigma(a) \in Q$ and hence $t = N_{L/F}(a) \in -Q \cap F = -P$ which is a contradiction. ▮

Thanks are due to A. Wadsworth for his help by proving (b). Next, we want to answer the above question for extensions L/F where F and L are local number fields or finite extensions of the rational function field $R(X)$, R real closed, by applying the results on scaled trace forms.

THEOREM 4. *Let F be a local number field. Let L/F be an extension of degree m and let φ be a form over F with* $\dim \varphi = mn$, $n \geqslant 2$.

(a) *Let m be odd and $n = 2$. Then there exists a form ρ over L with* $\dim \rho = n$ *and* $\mathrm{Tr}_{L/F}(\rho) \cong \varphi$ *unless F is nondyadic and* $\varphi = (0, 1, -1)$.

(b) *Let m be even or $n \geqslant 3$. Then there exists a form ρ over L with* $\dim \rho = n$ *and* $\mathrm{Tr}_{L/F}(\rho) \cong \varphi$ *if and only if* $\operatorname{dis} \varphi \in \Lambda_{L/F}$.

Proof. Let m be even and $\varphi = (0, b, e)$ where $e = \pm 1$, $b \in \Lambda_{L/F}$. By [K, 1.1(iv)], there exists some $a \in L^*$ such that $\langle L_a \rangle = 0$. If $b \notin F^{*2}$ or $\varphi = (0, 1, 1)$ then by Lemma 2 there exists some $a' \in L^*$ such that $\langle L_{a'} \rangle = (0, b, e)$. Then $\mathrm{Tr}_{L/F}(\langle a' \rangle + (n-1)\langle a \rangle) \cong \varphi$. Let $\varphi = (0, 1, -1)$. By Lemma 5, there exists some $a' \in L^*$ such that $\langle L_{a'} \rangle = (0, c, -1)$ for some $c \in F^*$. By Lemma 2, there exists some $a'' \in L^*$ such that $\langle L_{a''} \rangle = (0, c, 1)$. Then $\mathrm{Tr}_{L/F}(\langle a', a'' \rangle + (n-2)\langle a \rangle) \cong \varphi$. If m is odd the proof is similar and left to the reader. Use Lemma 1 if $m = 2$ and F is nondyadic. ▮

THEOREM 5. *Let F be a field with* $\operatorname{char} F \neq 2$ *such that for every finite extension K/F, $I^2(K)$ is torsion free and K satisfies SAP if K is formally real. Let L/F be a separable extension of degree m. Let φ be a form over F with* $\dim \varphi = nm$. *Then there exists a form ρ over L with* $\dim \rho = n$ *and* $\mathrm{Tr}_{L/F}(\rho) \cong \varphi$ *if and only if* $|\mathrm{sgn}_P \varphi| \leqslant n \# E_{L/F}(P)$ *for all* $P \in X_F$.

Proof. The condition for the signature values of φ is necessary by [S, 1, 3.4.5]. Let φ be a form over F with $\dim \varphi = nm$ such that $|\mathrm{sgn}_P \varphi| \leqslant n \# E_{L/F}(P)$ for all $P \in X_F$. Since F satisfies the property (ED) by [P, W, Theorem 2], there exist forms φ_i over F with $\dim \varphi_i = m$ and $|\mathrm{sgn}_P \varphi_i| \leqslant \# E_{L/F}(P)$ for all $P \in X_F$ such that $\varphi \cong \varphi_1 + \cdots + \varphi_n$. By [K, 3.6] there exist $a_i \in L^*$ such that $\mathrm{Tr}_{L/F}(\langle a_i \rangle) \cong \varphi_i$. Thus $\mathrm{Tr}_{L/F}(\langle a_1, ..., a_n \rangle) \cong \varphi$. ▮

Note that if F is a finite extension of $R(X)$, R real closed, then F satisfies the above conditions. If L/F is not separable, replace $\mathrm{Tr}_{L/F}$ by some linear form $s \neq 0$. Then the corresponding result holds for L/F as well. Finally, we want to mention that Theorem 2 can be generalized as follows:

THEOREM 6. *Let F be a number field or a finite extension of $R(X)$, R real closed, and let $m, n, r \in \mathbb{N}$. Let $\varphi_1, ..., \varphi_r$ be forms over F with* $\dim \varphi_i = mn$ *for $i = 1, ..., r$. Then there exist an extension L/F of degree m and forms $\rho_1, ..., \rho_r$ over L with* $\dim \rho_i = n$ *such that* $\mathrm{Tr}_{L/F}(\rho_i) \cong \varphi_i$ *for $i = 1, ..., r$.*

Proof. Let F be a number field. Let $m = 2^s l$. We proceed by induction on s. If $s = 0$, use Theorem 3 and the arguments pointed out in the proof of Proposition 3. Let $s \geqslant 1$. By induction hypothesis, there exist an extension M/F of degree $2^{s-1} l$ and ψ_i over M with $\dim \psi_i = 2n$ and $\mathrm{Tr}_{L/F}(\psi_i) \cong \varphi_i$ for $i = 1, ..., r$. By [K, 4.2], there exists a sum of squares $d \in M^* - M^{*2}$ such that $\operatorname{dis} \psi_i \in \Lambda_{L/M}$ for $L = M(\sqrt{d})$, $i = 1, ..., r$. Then by Proposition 6, there exist ρ_i over L with $\dim \rho_i = n$ such that $\mathrm{Tr}_{L/M}(\rho_i) \cong \psi_i$; that is $\mathrm{Tr}_{L/F}(\rho_i) \cong \varphi_i$ for $i = 1, ..., r$.

If F is a finite extension of $R(X)$, R real closed, choose some extension L/F of degree m such that $\# E_{L/F}(P) = m$ for all $P \in X_F$. Such an extension exists, since F is hilbertian (see [K, S, proof of Corollary 2]). Then apply Theorem 5. ∎

REMARK 5. *If the $\varphi_1, ..., \varphi_r$ in Theorem 6 are torsion forms, then we may assume $\rho_1, ..., \rho_r$ to be torsion forms over L. Moreover, if n is even we may assume that L is not formally real.*

Proof. The above proof can easily be modified to prove these statements. Details are left to the reader. ∎

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